Volume Renormalization.

- Conformal manifolds.

→ Consider a Riemannian manifold (M, g).
We say a metric $\hat{g}$ on M is conf. rel. to g if $\exists$ a smooth $\omega \ni \hat{g} = e^{2\omega} g$.

$$\hat{g} \in [g] \iff \exists \omega \ni \hat{g} = e^{2\omega} g.$$

$(M, [g])$ is called a conf. mfd.

*. Example: Consider the ball model of hyp. space $M = \mathbb{B}^{n+1}$ with $g^+ = \frac{4}{(1-|x|^2)^2} g_E$

- $Ric(g^+) = -n g^+$
- $R_{g^+} = -n(n+1)$

! g^+ cannot be extended to $\partial M = \mathbb{S}^n$ and neither does it define a metric there.

But $(1-|x|^2)^2 g^+ = 4 g_E$ extends to ∂M.

$\left(\frac{1-|x|}{1+|x|}\right)^2 g^+$ also extends to ∂M.

Call $r = \frac{1-|x|}{1+|x|}$ then $r^2 g^+$ and any $r^2 \Omega^2 g^+$ $(\Omega > 0)$ extend to ∂M

$$g = r^2 g^+ \rightsquigarrow h = g|_{T\partial M}$$
$$\hat{g} = r^2 \Omega^2 g^+ \rightsquigarrow \hat{h} = \hat{g}|_{T\partial M}$$

$(\mathbb{B}^{n+1}, g^+)$ determines $(\mathbb{S}^n, [h])$

$$\mathrm{Isom}(\mathbb{B}^{n+1}) \cong \mathrm{Conf}(\mathbb{S}^n)$$

★ Conformally Compact Einstein manifolds.

- Let M^{n+1} be a smooth mfd w/ bdry $\partial M = \Sigma^n$.
  - → We say a fxn r (smooth enough) is a defining fx^n for ∂M if
$$\begin{cases} r > 0 & \text{in } \mathring{M} \\ r = 0 & \text{on } \partial M \\ dr \neq 0 & \text{on } \partial M \end{cases}$$
  - → A Riemannian metric g^+ on $\mathring{M}$ is called a conformally compact metric if $(M, r^2 g^+)$ is a compact Riem. mfd.
- C.C. mfd $(\mathring{M}, g^+)$ carries a well defined conformal structure on its bdry if $h = r^2 g^+|_{T\partial M}$, $(\partial M, [h])$.
  - → We call $(\partial M, [h])$ the conformal infty of $(\mathring{M}, g^+)$.

- A. Conf. Compct. Einstein metric is a CC metric satisfying

$$\mathrm{Ric}(g^+) = -n g^+.$$

→ A. C.C. metric is called Asymp. Hyp. if for a choice of representative $h \in [h]$, $\exists\, r$ $\ni$ $|dr|_{r^2g^+} = 1$ $\quad h = r^2 g^+|_{\partial M}$

$$\underline{R^+_{ijkl}} = -|dr|_{r^2g^+} \left(g_{ik} g_{jl} - g_{il} g_{jk} \right) + \mathcal{O}(r^{-3})$$

[Graham-Lee '91] If $(\mathring{M}, g^+)$ is A.H. then a choice $h \in [h]$ on Σ uniquely determines a defining fx^n r (called spl. def fx^n) and an id^n of a nbd of Σ in M with $\Sigma \times [0, \varepsilon)$. s.t.

$$g^+ = \frac{1}{r^2} \left(dr^2 + h_r \right)$$

where h_r is a 1-par. family of metrics.

☆ A CCE metric is A.H.

[Graham '00] If g^+ is CCE and r is a spl defining fx^n, then

$$g^+ = \frac{1}{r^2} \left(dr^2 + h_r \right)$$

where

$$h_r = h + h^{(2)} r^2 + \underset{\text{even pwrs}}{\cdots} + h^{(n-1)} r^{n-1}$$
$$+ k\, r^n \ln r$$
$$+ h^{(n)} r^n + o(r^n)$$

- $h^{(j)}$ for $j < n$ are locally formally determined
- k is locally formally determined
- $tf(h^{(n)})$ is formally undetermined
- For n: odd $\quad k = 0$

  $\operatorname{tr} h^{(n)} = 0.$
- For n: even. $\operatorname{tr} h^{(n)}$ is locally formally det.

- $(\mathbb{B}^{n+1}, g^+)$ $\qquad r = \dfrac{1-|x|}{1+|x|} \sim$ is spl.

$$g^+ = \frac{1}{r^2}\left(dr^2 + \left(\frac{1-r^2}{2}\right)^2 h \right)$$

h is the round met. on $\mathbb{S}^n$.

$$g^+ = \frac{1}{r^2}\left(dr^2 + h_r\right)$$

$$dV_{g^+} = r^{-n-1} \underbrace{\sqrt{\frac{\det h_r}{\det h}}}\, dr\, dV_h$$

$$= r^{-n-1} \sqrt{\det h_r}\, dr\, dV_{h_r} = r^{-n-1} \cdot \sqrt{\frac{\det h_r}{\det h}}\, dr\, dV_h$$

$$\Rightarrow dV_{g^+} = v_0\, r^{-n-1} + v_2\, r^{-n+1} + \underset{\text{even steps}}{\cdots} + v_n\, r^{-1} + \cdots$$

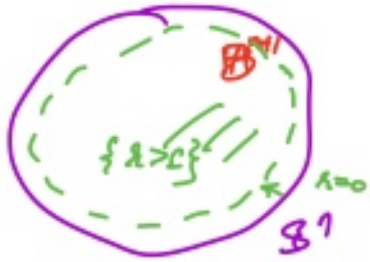
$v_n = 0$ for odd n.

$(\mathbb{B}^{n+1}, g^+)$ $\qquad g^+ = \frac{1}{r^2}(dr + h_r)$

in g^+. Vol. of $\mathbb{B}^{n+1}$ is infinite.

→ Renormalization of Volume.

Consider $Vol_{g^+}\{r > \varepsilon\}$



$$Vol_{g^+}\{r>\varepsilon\} = \underline{c_0}\,\varepsilon^{-n} + c_2\varepsilon^{-n+2} + \cdots \text{ even steps } + c_{n-1}\varepsilon^{-1} + \mathcal{E}\ln\frac{1}{\varepsilon} + V + o(1)$$

[Henningson-Skenderis 98] [Graham 00]

$\mathcal{E}$ is indep. of choice of h if n: even

V " " " " if n: odd.

*. For $(\mathbb{B}^{n+1}, g^+)$ $\quad g^+$ in normal form

$$dV_{g^+} = r^{-n-1}(1-r^2)^n\, dr\, dV_h.$$

$$2^{-n} Vol_{\mathring{g}}(S^n) \times \int_\varepsilon^1 r^{-n-1}(1-r^2)^n\, dr = \int_\varepsilon^1 dV_{g^+} \quad \text{gives}$$

$$\mathcal{E} = (-1)^{\frac{n}{2}} \frac{2\pi^{\frac{n}{2}}}{(\frac{n}{2})!} \qquad n: \text{even}$$

$$V = (-1)^{\frac{n+1}{2}} \frac{\pi^{\frac{n+2}{2}}}{\Gamma(\frac{n+2}{2})} \qquad n: \text{odd}.$$

* Singular Yamabe metrics.

- Let (M^{n+1}, g) be a Riem. mfd w/ bdry $\partial M = \Sigma$. set $h = g|_{T\Sigma}$.

→ A singular Yamabe metric on $\mathring{M}$ is a $g^+ = u^{-2} g$ s.t. $R_{g^+} = -n(n+1)$ for some smooth (enough) u.

$\exp : [0,\delta) \times \Sigma \longrightarrow M$ is a diff. onto a nbd of Σ in M.

under this idⁿ, g takes the form

$$g = dr^2 + h_r.$$

h_r : 1-par. family of metrics

$$h_r|_{r=0} = h$$

[Andersson - Chruściel - Friedrich '92]
[Graham '17]

The solⁿ u to $R_{u^{-2}g} = -n(n+1)$ expands asymptotically as

$$u = r + u^{(2)} r^2 + \cdots + u^{(n+1)} r^{n+1} + \mathcal{L}\, r^{n+2} \ln r + \cdots$$

! if $\hat{g} = \Omega^2 g$ then $\hat{\mathcal{L}} = (\Omega|_\Sigma)^{-n-1} \mathcal{L}$.

$$g^+ = u^{-2} g$$

$$dV_{g^+} = u^{-n-1}\, dV_g = u^{-n-1}\sqrt{\frac{\det h_r}{\det h}}\, dr\, dV_h$$

$$\Rightarrow dV_{g^+} = r^{-n-1}\left(1 + v^{(1)} r + v^{(2)} r^2 + \cdots + v^{(n)} r^n + O(r^{n+1}\ln r)\right)$$

$Vol_{g^+}(\mathring{M})$ is infinite!

Considering the volume of the region $\{r > \varepsilon\}$ we have

$$Vol_{g^+}\{r>\varepsilon\} = c_0\varepsilon^{-n} + c_1\varepsilon^{-n+1} + \cdots + c_{n-1}\varepsilon^{-1} + \mathcal{E}\ln\frac{1}{\varepsilon} + V + o(1)$$

[Graham '17] $\quad \mathcal{E}$ is indep. of choice of g

$$n=2 \qquad \mathcal{E} = \frac{1}{4}\int_\Sigma \left(|\mathring{\mathrm{II}}|^2 - R_h\right) dV_h$$

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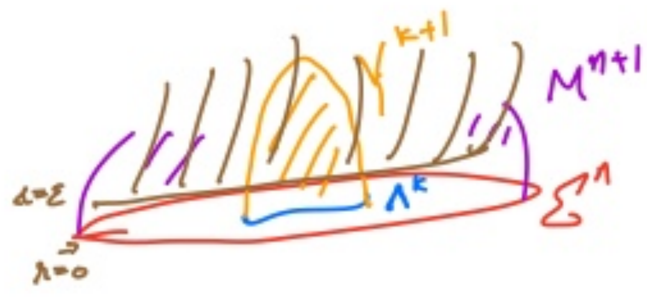
* Graham-Witten use the CCE picture to construct extrinsic conf. inv. of Λ embedded in $(\Sigma, [h])$.



Y: min. in M.

$Ar\{Y \cap \{r>\varepsilon\}\}$ expands in even steps.

$$= \underline{\qquad\qquad} + K\ln\frac{1}{\varepsilon} + A + o(1)$$